Hom and Tensor

The localization can also be expressed as a tensor product. We'll give a brief intro to tensor and Hom (which is "adjoint" to tensor: $\mathrm{Hom}(Y \otimes X, Z) \cong \mathrm{Hom}(Y, \mathrm{Hom}(X,Z))$), but you can read more in the appendix.

Hom

Def: If M, N are R-modules, then $\mathrm{Hom}_R(M,N)$ is the R-module of homomorphisms $M \to N$.

Ex: $\mathrm{Hom}_R\left(\bigoplus_{i=1}^{n} R, N\right) \cong \bigoplus_{i=1}^{n} N$

Hom is a functor in each of its entries:

Fix an R-module M.

Then $\mathrm{Hom}(M, -)$ takes a map $A \to B$ to a map

$$\begin{tikzcd} M \arrow[r] \arrow[dr, dashed] & A \arrow[d] \\ & B \end{tikzcd} \qquad \mathrm{Hom}_R(M,A) \longrightarrow \mathrm{Hom}_R(M,B).$$

$\mathrm{Hom}(M,-)$ is <u>left-exact</u>, i.e. if

$0 \to A \to B \to C$ is exact, then so is

$$0 \to \mathrm{Hom}(M,A) \longrightarrow \mathrm{Hom}(M,B) \longrightarrow \mathrm{Hom}(M,C).$$

$\mathrm{Hom}(-, M)$ is also a functor, but given $A \to B$,

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dashed] & B \arrow[d] \\ & M \end{tikzcd}$$

we get a map $\mathrm{Hom}(B,M) \to \mathrm{Hom}(A,M)$.

So it "reverses arrows." i.e. it's a <u>contravariant</u> functor.

If $A \to B \to C \to 0$ is exact, then

$$0 \to \mathrm{Hom}(C,M) \to \mathrm{Hom}(B,N) \to \mathrm{Hom}(A,N)$$ is exact.

Tensor products

If M and N are R-modules,

$M \otimes_R N$ is the R-module generated by elts of the form

$m \otimes n$, w/ $m \in M$, $n \in N$

s.t. it satisfies the following relations:

- If $r \in R$, $(rm) \otimes n = r(m \otimes n) = m \otimes (rn)$
- $(m+m') \otimes n = m \otimes n + m' \otimes n$ and
  $m \otimes (n+n') = m \otimes n + m \otimes n'$.

<u>Caution:</u> In general, elts look like finite sums $\sum_i m_i \otimes n_i$. It's usually difficult to tell if two given elts are equal. In fact, it's often a question what the minimum number of simple tensors required to express an element is. See "border rank."

<u>Ex:</u>

1.) $R \otimes_R M \cong M \cong M \otimes_R R$ and $M \otimes_R N \cong N \otimes_R M$

2.) $R[x_1,\dots,x_m] \otimes_R R[x_{m+1},\dots,x_n] \cong R[x_1,\dots,x_n]$

3.) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[x] \cong \mathbb{Q}[x]$.

4.) $I, J \subseteq R$ ideals $\Rightarrow$ $R/I \otimes_R R/J \cong R/(I+J)$

5.) If M is an R-module, S an R-algebra, then $S \otimes_R M$ is an S-module: $s(t \otimes m) = st \otimes m$.

Universal property

Note that $M \times N \to M \otimes N$ defined $m \times n \mapsto m \otimes n$ (as a function, not a morphism) is bilinear over R. In fact, if $f: M \times N \to P$ is bilinear, $\exists$ a unique morphism $\bar{f}: M \otimes N \to P$ s.t.

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "f"'] & M \otimes N \arrow[d, dashed, "\bar{f}"] \\ & P \end{tikzcd}$$

commutes.

This is called the universal property of tensor product.

Tensor product is a functor, and it is right-exact.

i.e. if

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is exact, then so is

$$A \otimes_R M \xrightarrow[\alpha \otimes id]{} B \otimes_R M \xrightarrow[\beta \otimes id]{} C \otimes_R M \to 0.$$

Geometric context: If R and S are coordinate rings, then $R \otimes S$ is the coordinte ring of the product of the two corresponding varieties.

Localization as tensor product.

We can describe the localization of a module by first localizing the ring and then tensoring:

Lemma: The map $R[U^{-1}] \otimes_R M \longrightarrow M[U^{-1}]$ defined $\frac{r}{u} \otimes m \longmapsto \frac{rm}{u}$ is an isomorphism.

Pf: We'll construct an inverse:

$$\varphi : M[U^{-1}] \longrightarrow R[U^{-1}] \otimes_R M \quad \text{defined}$$

$$\frac{m}{u} \longmapsto \frac{1}{u} \otimes m.$$

Why is this well-defined?

If $\frac{m}{u} = \frac{m'}{u'}$, then $vu'm = vum'$, some $v \in U$.

$$\Rightarrow vu' \otimes m = vu \otimes m' \Rightarrow \frac{1}{u} \otimes m = \frac{1}{u'} \otimes m'. \quad \square$$

Since we've written localization as a tensor product, we know that it is right-exact. In fact, it also preserves injections, and thus preserves exact sequences! This is called "flatness":

Def: An R-module F is <u>flat</u> if for every injective R-module map $M \to N$, $F \otimes M \to F \otimes N$ is injective as well.

Prop: $R[U^{-1}]$ is flat as an R-module. Thus, localization preserves exact sequences.

Pf. Assume $\varphi: M' \to M$ is an injection of R-modules.

We want to show $R[U^{-1}] \otimes_R M' \to R[U^{-1}] \otimes_R M$ is injective.

i.e. $M'[U^{-1}] \to M[U^{-1}]$.

If $\frac{m'}{u} \mapsto \frac{\varphi(m')}{u} = 0$, then $v\varphi(m') = 0$, some $v \in U$

$$\Rightarrow \varphi(vm') = 0 \Rightarrow vm' = 0$$

$$\Rightarrow \frac{m'}{u} = 0. \square$$

As previously mentioned, there are many properties of modules and rings that we can check by "checking locally".

Geometrically, for instance, we can check if a variety is smooth by checking smoothness at each point. Or, we can determine if a function on a variety is zero by checking whether it is zero at each point.

The algebraic analogue is the following:

Lemma: R a ring, M an R-module.

a.) If $a \in M$, then $a = 0 \iff \frac{a}{1} = 0$ in $M_{\mathfrak{m}}$ for each max'l ideal $\mathfrak{m} \subseteq R$

b.) $M = 0 \iff M_{\mathfrak{m}}$ $\forall$ max ideal $\mathfrak{m} \subseteq R$.

Pf: a.) $\frac{a}{1} = 0$ in $M_{\mathfrak{m}} \iff$ the annihilator I of a (i.e. the ideal $I \subseteq R$ s.t. $ra = 0$ $\forall$ $r \in I$) is not contained in $\mathfrak{m}$.

Thus $\frac{a}{1} = 0 \iff I = R \iff a = 0$ in M.

b.) $M = 0 \iff a = 0$ $\forall$ $a \in M \iff a/_1 = 0$ in all $M_{\mathfrak{m}}$. $\square$

Another thing we can check locally is injectivity and surjectivity:

Cor: If $\varphi: M \to N$ is a map of R-modules, then φ is injective (resp. surjective) iff $\varphi_{\mathfrak{m}}: M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is $\forall$ max'l ideals $\mathfrak{m}$.

Pf: One direction follows from flatness.

If $\ker(\varphi_m) = (\ker\varphi)_m = 0 \ \forall m$, then $\ker\varphi = 0$.

Similarly w/ cokernel. $\square$

Radical ideals

Recall that the complement of a prime ideal must be multiplicative. The converse doesn't hold:

Ex: The complement of $\{1, x, x^2, \dots\}$ in $k[x]$ isn't an ideal: $x^2+x+1 - (x+1) = x^2$.

However, the ideals that are max'l with respect to not meeting a multiplicative set are prime:

Suppose $I \subseteq R$ doesn't meet U, and if $J \supseteq I$ doesn't meet U then $J = I$.

Thus, $IR[U^{-1}]$ must be max'l (since any ideal in R meeting U will generate the unit ideal in $R[U^{-1}]$.)

Thus the preimage of $IR[U^{-1}]$ in R, P, is prime. But $P \supseteq I$, so $I = P$.

In particular, if $I \subseteq R$ an ideal, and $a \notin I$ s.t.

$a^n \notin I$ for all n, then there's some prime P s.t. $a \notin P \supseteq I$.

Conversely, if $a^n \in I$ for some n, then for all primes P containing I, $a \in P$. That is:

Cor: If $I \subseteq R$ is an ideal, then

$$\{f \mid f^n \in I \text{ for some } n\} = \bigcap_{\substack{P \supseteq I \\ \text{prime}}} P.$$

Def: The set above, $\{f \mid f^n \in I, \text{ some } n\}$ is called the <u>radical</u> of I, denoted $\operatorname{rad} I$ or $\sqrt{I}$.
Note that this is an ideal as it is the intersection of ideals.

The radical of (0) is the set of <u>nilpotent</u> <u>elements</u>, also called the <u>nilradical</u>.

Cor: $\operatorname{rad}(0)$ is the intersection of all of the primes of R.

Def: A ring R is <u>reduced</u> if $\operatorname{rad}(0) = (0)$.

Ex: $k[x]/(x^2)$ is <u>not</u> reduced. $\operatorname{rad}(0) = (x)$.

Note: The ideal of nilpotents is not always prime itself!

e.g. in $\mathbb{Z}/12\mathbb{Z}$, $\mathrm{rad}(0) = (\overline{6})$, which is not prime.

More generally, if R is a UFD and $f_1, \ldots, f_n$ irreducible elts generating distinct ideals, let $g = f_1^{k_1} \cdots f_n^{k_n}$, k_i positive integers.

Then the nilradical of $R/(g)$ is $(f_1 f_2 \cdots f_n)$.